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Quantum dynamics of parasupersymmetric and shape-invariant coupled systems

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Abstract
A class of bound-state problems which represents the coupling of a three-level atom with a two-dimensional system involving two shape-invariant potentials was introduced in a previous paper. In this paper, we considered second-order parasupersymmetric quantum-mechanical models and two possible kinds for the coupling Hamiltonian (linear and nonlinear in the potential ladder operators). In the present paper, using an algebraic formulation for shape-invariant potential systems, we study the quantum dynamics of these coupled systems and obtain the temporal behaviour of some dynamical variables related with the atom and the coupling potentials. An application is given for a couple of shape-invariant potentials (Pöschl–Teller + self-similar potentials).

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1. Introduction

Supersymmetric quantum mechanics [1] deals with the partner Hamiltonians

$$\hat{H}_- = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(-)}(x) = \hbar\Omega\hat{A}^\dagger\hat{A} \qquad \text{and} \qquad \hat{H}_+ = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(+)}(x) = \hbar\Omega\hat{A}\hat{A}^\dagger \tag{1}$$

which, in one dimension, can be written in terms of the operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) + \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}\right) \qquad \text{and} \qquad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) - \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}\right) \tag{2}$$

where $\hbar\Omega$ is a constant energy scale factor, introduced so that the operators $\hat{A}$ and $\hat{A}^\dagger$ are dimensionless. The function $W(x)$ is the superpotential which is related to the partner potentials $V^{(\pm)}(x)$ via

$$V^{(\pm)}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2M}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{3}$$

A number of such pairs of Hamiltonians $\hat{H}_{\pm}$ share the integrability condition $\hat{A}(a_1)\hat{A}^{\dagger}(a_1) = \hat{A}^{\dagger}(a_2)\hat{A}(a_2) + R(a_1)$ called shape invariance [2] where the parameter a_2 of the Hamiltonian is a function of its parameter a_1 and the remainder $R(a_1)$ is independent of the dynamical variables. In the cases studied so far the parameters a_1 and a_2 are either related by a translation [3, 4] or a scaling [5–9]. Supersymmetric quantum mechanics together with the shape invariance concept represents an elegant and powerful technique for obtaining closed analytic solutions for a set of potential systems with application in bound-state problems common in many areas of the physics [10]. Solvable models in quantum theory are so rare that they are worth studying in their own right. Even simplified they provide a clear understanding of physical phenomena involved and can be useful in controlling various approximations indispensable for the treating of more realistic cases.

In earlier publications [11–15] we introduced a class of shape-invariant coupled-channel problems which generalize the Jaynes–Cummings Hamiltonian [16], a simple model which is extensively used with success in quantum optics [17]. In this series of two papers, we extend the study that we began in our previous publications introducing a class of second-order parasupersymmetric coupled-channel problems consisting of a two-dimensional shape-invariant system interacting with a three-level atom or molecule. Parasupersymmetry is a natural extension of supersymmetric quantum mechanics introduced in order to include, in an elegant way, symmetry between bosonic and parafermionic degrees of freedom in a given system. In the first paper of this series [18], we presented and discussed models and, by using the algebraic formulation of shape-invariant systems of [8], obtained explicit expressions for the eigenvalues and eigenstates of the coupled system for each type of level configuration of the three-level system, considering two possible forms of coupling: a linear and another nonlinear in the potential ladder operators. The dynamics of this kind of coupled system is strongly dependent on the initial conditions, i.e., on the states in which the parasupersymmetric and shape-invariant potential systems and the atom are prepared at the beginning. We study in this second paper of the series quantum dynamics of the models, analysing the evolution of some observables and investigating their impact on the coupled system. The organization of the paper is as follows: in section 2 we present the Hamiltonian of the coupled system for each type of configuration of the three-level atom and obtain its time-evolution operator and the density operator; in section 3 we obtain the temporal behaviour of the atomic level occupation and level transition probabilities as well as the quantum states probability distribution and the intensity of each coupling potential; in section 4 we apply our generalized results to a pair of shape-invariant potentials (namely a Pöschl–Teller potential plus a self-similar potential) discussing the relevant aspects of the time behaviour of each observable. Conclusion and brief remarks are given in section 5.

2. Parasupersymmetric and shape-invariant coupled systems

2.1. Hamiltonian

In this study, we treat three subsystems consisting of a single three-level atom or molecule simultaneously interacting with two shape-invariant potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$. If we assume that the eigenstates $|j\rangle_{\mathrm{A}}$ of the Hamiltonian $\hat{H}_{\mathrm{A}}$ of a non-interacting three-level atom $\hat{H}_{\mathrm{A}}|j\rangle_{\mathrm{A}} = \hbar\omega_j|j\rangle_{\mathrm{A}}$ (with $j = 1, 2, 3$) form a basis, then it is possible to write the free atom Hamiltonian in the form

$$\hat{H}_{\mathrm{A}} = \hbar \sum_{j=1}^{3} \omega_j \hat{\sigma}_{jj} \tag{4}$$

where the projection operator $\hat{\sigma}_{jj} \equiv |j\rangle_{\rm AA}\langle j|$ describes the population of the level j whose energy is $\hbar\omega_j$. The atomic-transition operators $\hat{\sigma}_{jk} \equiv |j\rangle_{\rm AA}\langle k|$ from the level k to the level j (with $j, k = 1, 2, 3$ and $j \neq k$) and the projection $\hat{\sigma}_{jj}$ operators obey the relations

$$\hat{\sigma}_{jk}\hat{\sigma}_{rs} = \delta_{kr}\hat{\sigma}_{js} \qquad \text{and} \qquad \sum_{j=1}^{3}\hat{\sigma}_{jj} = \hat{\mathbb{1}} \tag{5}$$

together with the commutation and anticommutation relations

$$[\hat{\sigma}_{jk}, \hat{\sigma}_{rs}] = \delta_{kr}\hat{\sigma}_{js} - \delta_{js}\hat{\sigma}_{rk}, \qquad \{\hat{\sigma}_{jk}, \hat{\sigma}_{rs}\} = \delta_{kr}\hat{\sigma}_{js} + \delta_{js}\hat{\sigma}_{rk}, \tag{6}$$

characteristic of the generators of the group $SU(3)$. By assuming a three dimension spinor representation χ_j for the atomic eigenstates

$$\chi_1 \equiv \langle\chi|1\rangle_{\rm A} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \chi_2 \equiv \langle\chi|2\rangle_{\rm A} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \chi_3 \equiv \langle\chi|3\rangle_{\rm A} = \begin{bmatrix}0\\0\\1\end{bmatrix} \tag{7}$$

it is straightforward to verify that in such representation the transition and projection operators must be represented by 3×3 matrices obtained from the product $\hat{\sigma}_{jk} = \chi_j\chi_k^{\dagger}$.

There are three distinct energy level configurations for a three-level system known as Ξ or cascade configuration, Λ or Raman configuration and V configuration [19]. The total Hamiltonian describing a system with a three-level atom coupled to two shape-invariant potentials may be written as $\hat{H}_{\rm T}^{(\rm X)} = \hat{H}_{\rm A} + \hat{H}_{\rm P}^{(\rm X)} + \hat{H}_{\xi}^{(\rm X)}$ where $\hat{H}_{\rm P}^{(\rm X)}$ is the Hamiltonian, related with the shape-invariant potential systems, and $\hat{H}_{\xi}^{(\rm X)}$ is the atom–potentials interaction Hamiltonian. In a parasupersymmetric formulation, both the Hamiltonians $\hat{H}_{\rm P}^{(\rm X)}$ and $\hat{H}_{\xi}^{(\rm X)}$ depend on the possible types of atomic level configurations (X = Ξ, Λ, V). We construct these Hamiltonians based on the requirements imposed by the second-order multidimensional parasuperalgebra [20, 21] and by the dipole and rotating wave approximations. For each type of configuration we also consider two possible forms of interaction which correspond to the shape-invariant generalization of the ordinary or usual and intensity-dependent or nonlinear interaction forms used in quantum optics models [17], and specified here by assuming (ξ = U) and (ξ = N), respectively. The intensity-dependent interaction makes the enhancement of certain quantum effects possible that would be otherwise difficult to note within the realm of usual interaction model [13–15]. As additional basic statements of the model, we assume that each shape-invariant potential interacts with a couple of levels in such a way that direct transitions are allowed between atomic levels 1 and 2 and between levels 2 and 3, and forbidden between levels 1 and 3.

The algebraic formulation for shape-invariant systems presented in [8] can be applied in the Hamiltonian $\hat{H}_{\rm X} = \hat{H}_{\rm P}^{(\rm X)} + \hat{H}_{\xi}^{(\rm X)}$ by using the operators $\hat{B}_{+}^{(k)} \equiv \hat{A}_k^{\dagger}\hat{T}_k$ and $\hat{B}_{-}^{(k)} \equiv \hat{T}_k^{\dagger}\hat{A}_k$, defined with the introduction of the parameter translation operators $\hat{T}_1 \equiv \hat{T}_1\big(a_1^{(1)}\big)$ and $\hat{T}_2 \equiv \hat{T}_2\big(a_1^{(2)}\big)$ for each shape-invariant potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$. In these circumstances, the commutation relations which exhibit the shape invariance condition and the primary independence of the three subsystems (the atom and the two potentials), $\big[\hat{B}_{\mp}^{(k)}, \hat{B}_{\pm}^{(j)}\big] = \pm R_k\big(a_0^{(k)}\big)\delta_{kj}$, $\big[\hat{B}_{\pm}^{(k)}, \hat{\sigma}_{ij}\big] = 0$ and $\big[\hat{B}_{\pm}^{(k)}, \hat{B}_{\pm}^{(j)}\big] = 0$, are satisfied. Note that $R_1\big(a_n^{(1)}\big)$ and $R_2\big(a_n^{(2)}\big)$ are the remainders related with the potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$, respectively. With the basic model assumptions presented above the parasupersymmetric and the coupling potentials Hamiltonians $\hat{H}_{\rm X} = \hat{H}_{\rm P}^{(\rm X)} + \hat{H}_{\xi}^{(\rm X)}$ for the atomic level configuration X = Ξ, Λ, V

can be written in the algebraic formulation for shape-invariant systems in the final form $\hat{H}_{\rm X} = \hat{\mathcal{T}}_{\rm X}\hat{h}_{\rm X}\hat{\mathcal{T}}_{\rm X}^{\dagger}$ if we define the inclusive parameter translation operator

$$\hat{\mathcal{T}}_{\rm X} = \begin{cases} \hat{T}_1\hat{T}_2\hat{\sigma}_{11} + \hat{T}_1\hat{\sigma}_{22} + \hat{\sigma}_{33}, & {\rm X} = \Xi; \\ \hat{T}_1\hat{\sigma}_{11} + \hat{T}_1\hat{T}_2\hat{\sigma}_{22} + \hat{T}_2\hat{\sigma}_{33}, & {\rm X} = \Lambda; \\ \hat{T}_2\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{T}_1\hat{\sigma}_{33}, & {\rm X} = {\rm V}; \end{cases} \tag{8}$$

and decompose the Hamiltonian $\hat{h}_{\rm X}$ in $\hat{h}_{\rm X} = \hat{h}_{\rm P}^{({\rm X})} + \hat{h}_{\xi}^{({\rm X})}$ with

$$\hat{h}_{\rm P}^{({\rm X})} = \hbar\Omega \begin{cases} \big[\big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{11} + \big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{22} + \big(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{33}\big], & {\rm X} = \Xi; \\ \big[\big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{11} + \big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{22} + \big(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{33}\big], & {\rm X} = \Lambda; \\ \big[\big(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{11} + \big(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{22} + \big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{33}\big], & {\rm X} = {\rm V}; \end{cases} \tag{9}$$

where $\hat{\mathcal{H}}_{\pm}^{(k)} = \hat{B}_{\mp}^{(k)}\hat{B}_{\pm}^{(k)}$, with $k = 1, 2$. The Hamiltonian $\hat{h}_{\xi}^{({\rm X})}$ can be rewritten in the form $\hat{h}_{\xi}^{({\rm X})} = \hat{h}_{\Delta} + \hat{w}_{\xi}^{({\rm X})}$ where for usual interaction case

$$\hat{w}_{\rm U}^{({\rm X})} = \hbar g \begin{cases} \big(\hat{B}_-^{(1)}\hat{\sigma}_{23} + \hat{B}_+^{(1)}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\hat{\sigma}_{12} + \hat{B}_+^{(2)}\hat{\sigma}_{21}\big), & {\rm X} = \Xi; \\ \big(\hat{B}_-^{(1)}\hat{\sigma}_{23} + \hat{B}_+^{(1)}\hat{\sigma}_{32} + \hat{B}_+^{(2)}\hat{\sigma}_{12} + \hat{B}_-^{(2)}\hat{\sigma}_{21}\big), & {\rm X} = \Lambda; \\ \big(\hat{B}_+^{(1)}\hat{\sigma}_{23} + \hat{B}_-^{(1)}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\hat{\sigma}_{12} + \hat{B}_+^{(2)}\hat{\sigma}_{21}\big), & {\rm X} = {\rm V}; \end{cases} \tag{10}$$

while for a nonlinear interaction case one has

$$\hat{w}_{\rm N}^{({\rm X})} = \hbar g \begin{cases} \Big(\hat{B}_-^{(1)}\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\sigma}_{23} + \sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{B}_+^{(1)}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\sigma}_{12} + \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{B}_+^{(2)}\hat{\sigma}_{21}\Big), & {\rm X} = \Xi; \\ \Big(\hat{B}_-^{(1)}\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\sigma}_{23} + \sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{B}_+^{(1)}\hat{\sigma}_{32} + \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{B}_+^{(2)}\hat{\sigma}_{12} + \hat{B}_-^{(2)}\sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\sigma}_{21}\Big), & {\rm X} = \Lambda; \\ \Big(\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{B}_+^{(1)}\hat{\sigma}_{23} + \hat{B}_-^{(1)}\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\sigma}_{12} + \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{B}_+^{(2)}\hat{\sigma}_{21}\Big), & {\rm X} = {\rm V}; \end{cases} \tag{11}$$

where g is the real coupling constant strength. The detuning term is given by $\hat{h}_{\Delta} = \hbar\{\Delta_1\hat{\sigma}_{11} + \Delta_3\hat{\sigma}_{33}\}$, where Δ_1 and Δ_3 are the detuning parameters.

2.2. Time evolution operator and the state of the system

With the Hamiltonian $\hat{H}_{\rm X}$ presented above, the Schrödinger equation for the coupled system is given by

$$\hat{H}_{\rm X}\big|\Psi_{\xi}^{({\rm X})}(t)\big\rangle = {\rm i}\hbar\frac{\partial}{\partial t}\big|\Psi_{\xi}^{({\rm X})}(t)\big\rangle. \tag{12}$$

Now, if we introduce the algebraic intrinsic wave state $\big|\psi_{\xi}^{({\rm X})}(t)\big\rangle$ by $\big|\Psi_{\xi}^{({\rm X})}(t)\big\rangle = \hat{\mathcal{T}}_{\rm X}\big|\psi_{\xi}^{({\rm X})}(t)\big\rangle$ and use it in Schrödinger equation (12) we find

$$\hat{h}_{\rm X}\big|\psi_{\xi}^{({\rm X})}(t)\big\rangle = {\rm i}\hbar\frac{\partial}{\partial t}\big|\psi_{\xi}^{({\rm X})}(t)\big\rangle, \tag{13}$$

where we took into account the unitarity property $\hat{\mathcal{T}}_{\rm X}\hat{\mathcal{T}}_{\rm X}^{\dagger} = \hat{\mathcal{T}}_{\rm X}^{\dagger}\hat{\mathcal{T}}_{\rm X} = \hat{1}$. Individual terms of the coupled-system Hamiltonian satisfy the general commutation relations $\big[\hat{H}_{\rm A}, \hat{h}_{\rm P}^{({\rm X})}\big] = \big[\hat{h}_{\rm P}^{({\rm X})}, \hat{h}_{\Delta}\big] = \big[\hat{h}_{\rm P}^{({\rm X})}, \hat{w}_{\xi}^{({\rm X})}\big] = 0$ and, for the case when $\Delta_1 = \omega_2 - \omega_1$ and $\Delta_3 = \omega_2 - \omega_3$, it is also easy to verify that $\big[(\hat{H}_{\rm A} + \hat{h}_{\Delta}), \hat{w}_{\xi}^{({\rm X})}\big] = 0$. Therefore, in this resonant condition, if we write the algebraic intrinsic wave state $\big|\psi_{\xi}^{({\rm X})}(t)\big\rangle$ as

$$\big|\psi_{\xi}^{({\rm X})}(t)\big\rangle = \exp\big(-{\rm i}\hat{h}_0^{({\rm X})}t/\hbar\big)\big|\varphi_{\xi}^{({\rm X})}(t)\big\rangle \qquad \text{with} \qquad \hat{h}_0^{({\rm X})} = \hat{H}_{\rm A} + \hat{h}_{\Delta} + \hat{h}_{\rm P}^{({\rm X})} \tag{14}$$

Table 1. Expansion factors of the time-evolution operator $\hat{U}_{\mathrm{U}}^{(\mathrm{X})}(t,0)$.

Configuration	$\mathrm{X}=\Xi$	$\mathrm{X}=\Lambda$	$\mathrm{X}=\mathrm{V}$
$\hat{u}_{11}^{(\mathrm{X,U})}(t)$	$\hat{1}+f_{\mathrm{C}}\left(\hat{\mu}_{++}^{(\mathrm{U})}t\right)\hat{\mathcal{H}}_{+}^{(2)}$	$\hat{1}+f_{\mathrm{C}}\left(\hat{\mu}_{+-}^{(\mathrm{U})}t\right)\hat{\mathcal{H}}_{-}^{(2)}$	$\hat{1}+f_{\mathrm{C}}\left(\hat{\mu}_{-+}^{(\mathrm{U})}t\right)\hat{\mathcal{H}}_{+}^{(2)}$
$\hat{u}_{22}^{(\mathrm{X,U})}(t)$	$\cos\left(g\hat{\mu}_{+-}^{(\mathrm{U})}t\right)$	$\cos\left(g\hat{\mu}_{++}^{(\mathrm{U})}t\right)$	$\cos\left(g\hat{\mu}_{--}^{(\mathrm{U})}t\right)$
$\hat{u}_{33}^{(\mathrm{X,U})}(t)$	$\hat{1}+f_{\mathrm{C}}\left(\hat{\mu}_{--}^{(\mathrm{U})}t\right)\hat{\mathcal{H}}_{-}^{(1)}$	$\hat{1}+f_{\mathrm{C}}\left(\hat{\mu}_{-+}^{(\mathrm{U})}t\right)\hat{\mathcal{H}}_{-}^{(1)}$	$\hat{1}+f_{\mathrm{C}}\left(\hat{\mu}_{+-}^{(\mathrm{U})}t\right)\hat{\mathcal{H}}_{+}^{(1)}$
$\hat{u}_{13}^{(\mathrm{X,U})}(t)$	$f_{\mathrm{C}}\left(\hat{\mu}_{++}^{(\mathrm{U})}t\right)\hat{B}_{-}^{(2)}\hat{B}_{-}^{(1)}$	$f_{\mathrm{C}}\left(\hat{\mu}_{+-}^{(\mathrm{U})}t\right)\hat{B}_{+}^{(2)}\hat{B}_{-}^{(1)}$	$f_{\mathrm{C}}\left(\hat{\mu}_{-+}^{(\mathrm{U})}t\right)\hat{B}_{-}^{(2)}\hat{B}_{+}^{(1)}$
$\hat{u}_{12}^{(\mathrm{X,U})}(t)$	$f_{\mathrm{S}}\left(\hat{\mu}_{++}^{(\mathrm{U})}t\right)\hat{B}_{-}^{(2)}$	$f_{\mathrm{S}}\left(\hat{\mu}_{+-}^{(\mathrm{U})}t\right)\hat{B}_{+}^{(2)}$	$f_{\mathrm{S}}\left(\hat{\mu}_{-+}^{(\mathrm{U})}t\right)\hat{B}_{-}^{(2)}$
$\hat{u}_{23}^{(\mathrm{X,U})}(t)$	$f_{\mathrm{S}}\left(\hat{\mu}_{+-}^{(\mathrm{U})}t\right)\hat{B}_{-}^{(1)}$	$f_{\mathrm{S}}\left(\hat{\mu}_{++}^{(\mathrm{U})}t\right)\hat{B}_{-}^{(1)}$	$f_{\mathrm{S}}\left(\hat{\mu}_{--}^{(\mathrm{U})}t\right)\hat{B}_{+}^{(1)}$

Observation: $\hat{u}_{kj}^{(\mathrm{X,U})}(t)=\{\hat{u}_{jk}^{(\mathrm{X,U})}(t)\}^{\dagger}$.

and use it into (13) we obtain

$$\hat{w}_{\xi}^{(\mathrm{X})}\left|\varphi_{\xi}^{(\mathrm{X})}(t)\right\rangle=\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\varphi_{\xi}^{(\mathrm{X})}(t)\right\rangle. \tag{15}$$

Introducing the time-evolution operator $\hat{U}_{\xi}^{(\mathrm{X})}(t,0)$ such as $\left|\varphi_{\xi}^{(\mathrm{X})}(t)\right\rangle=\hat{U}_{\xi}^{(\mathrm{X})}(t,0)|\varphi(0)\rangle$, where $|\varphi(0)\rangle$ is an arbitrary initial state, and inserting this definition into equation (15), we find

$$\hat{w}_{\xi}^{(\mathrm{X})}\hat{U}_{\xi}^{(\mathrm{X})}(t,0)=\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{U}_{\xi}^{(\mathrm{X})}(t,0). \tag{16}$$

Since the Hamiltonian term $\hat{w}_{\xi}^{(\mathrm{X})}$ is time independent, the solution of (16) satisfying the initial condition $\hat{U}_{\xi}^{(\mathrm{X})}(0,0)=\hat{1}$ can be written formally as

$$\hat{U}_{\xi}^{(\mathrm{X})}(t,0)=\exp\left(-\mathrm{i}\hat{w}_{\xi}^{(\mathrm{X})}t/\hbar\right). \tag{17}$$

Taking into account the series expansion of the exponential function, expressions (10) and (11) for the Hamiltonian $\hat{w}_{\xi}^{(\mathrm{X})}$ and using the properties (5) and (6) of the $\hat{\sigma}_{jk}$-operators, it is possible, after a long and tedious calculation, to obtain the final expression for the time-evolution operator

$$\hat{U}_{\xi}^{(\mathrm{X})}(t,0)=\sum_{j=1}^{3}\sum_{k=1}^{3}\hat{u}_{jk}^{(\mathrm{X},\xi)}(t)\hat{\sigma}_{jk} \tag{18}$$

where the expansion factors $\hat{u}_{jk}^{(\mathrm{X},\xi)}(t)$ for the usual interaction case are given in table 1. In this case, the auxiliary functions introduced there are defined by

$$f_{\mathrm{C}}(\hat{\mu}t)=\left\{\frac{\cos(g\hat{\mu}t)-\hat{1}}{\hat{\mu}^{2}}\right\}\qquad\text{and}\qquad f_{\mathrm{S}}(\hat{\mu}t)=-\mathrm{i}\left\{\frac{\sin(g\hat{\mu}t)}{\hat{\mu}}\right\} \tag{19}$$

and their operator argument factors are given by

$$\hat{\mu}_{\pm\pm}^{(\mathrm{U})}=\sqrt{\hat{\mathcal{H}}_{\pm}^{(1)}+\hat{\mathcal{H}}_{\pm}^{(2)}}. \tag{20}$$

For any analytical function $f(x)$ it is easy to show that

$$\hat{B}_{\pm}^{(k)}f\left\{\hat{B}_{\mp}^{(k)}\hat{B}_{\pm}^{(k)}\right\}=f\left\{\hat{B}_{\pm}^{(k)}\hat{B}_{\mp}^{(k)}\right\}\hat{B}_{\pm}^{(k)}. \tag{21}$$

Using this property it is possible to obtain the expansion coefficients $\hat{u}_{jk}^{(\mathrm{X,N})}(t)$ of the time-evolution operator $\hat{U}_{\mathrm{N}}^{(\mathrm{X})}(t,0)$, for the nonlinear interaction case, from the expressions in

table 1 of the coefficients $\hat{u}_{jk}^{(\mathrm{X,U})}(t)$ in the usual interaction case. For that it is enough to do the replacement of the operators $\hat{B}_{-}^{(k)} \to \hat{B}_{-}^{(k)}\sqrt{\hat{\mathcal{H}}_{-}^{(k)}}$ and $\hat{B}_{+}^{(k)} \to \sqrt{\hat{\mathcal{H}}_{-}^{(k)}}\,\hat{B}_{+}^{(k)}$, which, consequently, imply the replacement

$$\hat{\mathcal{H}}_{-}^{(k)} = \hat{B}_{+}^{(k)}\hat{B}_{-}^{(k)} \longrightarrow \sqrt{\hat{\mathcal{H}}_{-}^{(k)}}\,\hat{B}_{+}^{(k)}\hat{B}_{-}^{(k)}\sqrt{\hat{\mathcal{H}}_{-}^{(k)}} = \left\{\hat{\mathcal{H}}_{-}^{(k)}\right\}^2$$

$$\hat{\mathcal{H}}_{+}^{(k)} = \hat{B}_{-}^{(k)}\hat{B}_{+}^{(k)} \longrightarrow \hat{B}_{-}^{(k)}\sqrt{\hat{\mathcal{H}}_{-}^{(k)}}\sqrt{\hat{\mathcal{H}}_{-}^{(k)}}\,\hat{B}_{+}^{(k)} = \left\{\hat{\mathcal{H}}_{+}^{(k)}\right\}^2$$

and thus define the operator argument factors for this case as

$$\hat{\mu}_{\pm\pm}^{(\mathrm{N})} = \sqrt{\left\{\hat{\mathcal{H}}_{\pm}^{(1)}\right\}^2 + \left\{\hat{\mathcal{H}}_{\pm}^{(2)}\right\}^2}. \tag{22}$$

Knowing the time-evolution operator for the coupled system, we can write the final expression for the algebraic intrinsic wave state of the system as

$$\left|\psi_{\xi}^{(\mathrm{X})}(t)\right\rangle = \exp\left(-\mathrm{i}\hat{h}_0^{(\mathrm{X})}t/\hbar\right)\hat{U}_{\xi}^{(\mathrm{X})}(t,0)|\varphi(0)\rangle, \tag{23}$$

which is still valid for any couple of shape-invariant potentials. We remark at this point that using the similarity transformation $\hat{H}_{\mathrm{X}} = \hat{\mathcal{T}}_{\mathrm{X}}\hat{h}_{\mathrm{X}}\hat{\mathcal{T}}_{\mathrm{X}}^{\dagger}$ associating the original Hamiltonian $\hat{H}_{\mathrm{X}}$ with the Hamiltonian $\hat{h}_{\mathrm{X}}$, and using the algebraic intrinsic wave state $\left|\psi_{\xi}^{(\mathrm{X})}(t)\right\rangle$, lead us to the time-evolution operator $\hat{U}_{\xi}^{(\mathrm{X})}(t,0)$ in terms of the operators $\hat{B}_{\pm}^{(1,2)}$ only, without the use of the parameter translation inclusive operator $\hat{\mathcal{T}}_{\mathrm{X}}$.

2.3. The density operator

A simple and elegant way of incorporating statistical distributions of initial conditions into quantum dynamics of the coupled system is to represent the state of the quantum system by using the Hermitian density operator, defined as $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$. At any time $t > 0$, the time evolution of the operator $\hat{\rho}(t)$ is given by the Liouville equation of motion $\mathrm{i}\hbar\,\mathrm{d}\hat{\rho}(t)/\mathrm{d}t = [\hat{H}(t), \hat{\rho}(t)]$, obtained from the density operator definition and the Hamiltonian $\hat{H}(t)$ of the system. On the other hand, knowledge of $\hat{\rho}(t)$ enables us to obtain the expectation value of any observable $\hat{O}$ through

$$\langle\hat{O}(t)\rangle = \frac{\mathrm{Tr}\{\hat{\rho}(t)\hat{O}\}}{\mathrm{Tr}\{\hat{\rho}(t)\}}. \tag{24}$$

To apply the density matrix formalism to the parasupersymmetric coupled problem we use the algebraic intrinsic state vector (23) to obtain

$$\hat{\rho}_{\xi}^{(\mathrm{X})}(t) = \exp\left(-\mathrm{i}\hat{h}_0^{(\mathrm{X})}t/\hbar\right)\hat{U}_{\xi}^{(\mathrm{X})}(t,0)\hat{\rho}_{\mathrm{o}}\hat{U}_{\xi}^{(\mathrm{X})\dagger}(t,0)\exp\left(-\mathrm{i}\hat{h}_0^{(\mathrm{X})}t/\hbar\right) \tag{25}$$

where $\hat{\rho}_{\mathrm{o}} = |\varphi(0)\rangle\langle\varphi(0)|$. In the analysis of the dynamics of the coupled system it is very instructive to assume that at time $t = 0$ its quantum state is uncorrelated, i.e., it is described for a pure state obtained as the direct product $|\varphi(0)\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi\rangle_{\mathrm{A}}$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ are the coupling potential initial states. In spite of its apparent simplicity, the coupled-system models based on the dipole and rotating wave approximations usually exhibit in their dynamics a quite complicated behaviour and fully quantum-mechanical effects. Many details of the dynamics of the system strongly depend on its initial condition, and in order to understand the global influence of the shape-invariant potentials on the system dynamics, in this study we consider at $t = 0$ both potentials to be in coherent states and the three-level atom to be in one of its eigenstates, here assumed as the state $|\psi\rangle_{\mathrm{A}} = |3\rangle_{\mathrm{A}}$.

In this point we remember that the nth excited state of the Hamiltonians $\hat{\mathcal{H}}_{\pm}^{(k)}$ for shape-invariant systems satisfies the eigenvalue equations [22]

$$\hat{\mathcal{H}}_{-}^{(k)}|n\rangle_k = e_n^{(k)}|n\rangle_k \qquad \text{and} \qquad \hat{\mathcal{H}}_{+}^{(k)}|n\rangle_k = \left[e_n^{(k)} + R_k\left(a_0^{(k)}\right)\right]|n\rangle_k, \tag{26}$$

with the correspondents eigenvalues given by $e_0^{(k)} = 0$ and

$$e_n^{(k)} = \sum_{j=1}^{n} R_k\left(a_j^{(k)}\right) \qquad \text{for} \quad n \geqslant 1. \tag{27}$$

In previous works [22, 23], we showed that the coherent states for shape-invariant systems with an infinite number of bound-state energy levels can be obtained in a generalized way by the expansion in the basis $\{|n\rangle_k; n = 1, 2, 3, \ldots\}$

$$\begin{aligned} \left|z; a_r^{(k)}\right\rangle &= \sum_{n=0}^{\infty} \frac{z^n}{h_n\left(a_r^{(k)}\right)} |n\rangle_k \qquad \text{with} \quad h_0\left(a_r^{(k)}\right) = 1, \\ h_n\left(a_r^{(k)}\right) &= \prod_{s=0}^{n-1} \left[\frac{\sqrt{e_n^{(k)} - e_s^{(k)}}}{\mathcal{Z}_{j+s}^{(k)}}\right], \qquad \text{for} \quad n \geqslant 1. \end{aligned} \tag{28}$$

In this expression $\mathcal{Z}_{j+s}^{(k)} = \hat{T}_k^s \mathcal{Z}_j^{(k)} \hat{T}_k^{\dagger s} = \mathcal{Z}^{(k)}\left(a_{1+s}^{(k)}, a_{2+s}^{(k)}, a_{3+s}^{(k)}, \ldots\right)$, with $\mathcal{Z}_j^{(k)} \equiv \mathcal{Z}^{(k)}\left(a_1^{(k)}, a_2^{(k)}, a_3^{(k)}, \ldots\right)$ being an arbitrary functional of the potential parameters.

With these assumptions the initial state of the coupled system is described by

$$|\varphi(0)\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} b_{n_1}^{(1)} b_{n_2}^{(2)} |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes |3\rangle_{\mathrm{A}} \tag{29}$$

where, by equation (28), $b_{n_k}^{(k)} = z_k^{n_k} / h_{n_k}^{(k)}\left(a_r^{(k)}\right) \in \mathbb{C}$, with $k = 1$ or 2.

Using the property (21) and expression (29) to get $\hat{\rho}_{\mathrm{o}}$ it is possible to show that the time-evolved density operator (25) can be explicitly expressed in the matrix form

$$\hat{\boldsymbol{\rho}}_{\xi}^{(\mathrm{X})}(t) = \frac{1}{\mathcal{N}} \begin{bmatrix} \left|\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right| & \left|\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right| & \left|\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right| \\ \left|\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right| & \left|\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right| & \left|\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right| \\ \left|\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right| & \left|\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right| & \left|\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right\rangle\left\langle\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right| \end{bmatrix} \tag{30}$$

where the factor

$$\mathcal{N} = \langle\varphi(0)|\varphi(0)\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \left|b_{n_1}^{(1)}\right|^2 \left|b_{n_2}^{(2)}\right|^2 \tag{31}$$

was introduced to have the normalization condition $\mathrm{Tr}\left\{\hat{\rho}_{\xi}^{(\mathrm{X})}(t)\right\} = 1$ satisfied. Using equations (14), (18), (23) and (29) we obtain the expressions for the states which compose the matrix elements of $\hat{\boldsymbol{\rho}}_{\xi}^{(\mathrm{X})}(t)$ as

$$\begin{aligned} \left|\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\right\rangle &= \exp\left\{-\mathrm{i}\Omega\hat{\mathcal{H}}_1^{(\mathrm{X})} t\right\} \hat{u}_{13}^{(\mathrm{X},\xi)}(t)|\psi_{1,2}\rangle \\ \left|\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\right\rangle &= \exp\left\{-\mathrm{i}\Omega\hat{\mathcal{H}}_2^{(\mathrm{X})} t\right\} \hat{u}_{23}^{(\mathrm{X},\xi)}(t)|\psi_{1,2}\rangle \\ \left|\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\right\rangle &= \exp\left\{-\mathrm{i}\Omega\hat{\mathcal{H}}_3^{(\mathrm{X})} t\right\} \hat{u}_{33}^{(\mathrm{X},\xi)}(t)|\psi_{1,2}\rangle \end{aligned} \tag{32}$$

where $|\psi_{1,2}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. At this point we write the parasupersymmetric Hamiltonian $\hat{h}_{\mathrm{P}}^{(\mathrm{X})}$ in the compact form

$$\hat{h}_{\mathrm{P}}^{(\mathrm{X})} = \hbar\Omega \sum_{j=1}^{3} \hat{\mathcal{H}}_j^{(\mathrm{X})} \hat{\sigma}_{jj} \tag{33}$$

where the terms $\hat{\mathcal{H}}_j^{(\mathrm{X})}$ for each type of configuration can be identified looking at equation (9).

3. Temporal behaviour of the quantum dynamical variables

3.1. The subsystems' reduced density matrix

In order to explore the dynamics of each subsystem which make up the coupled system we must calculate from the density operator $\hat{\rho}_{\xi}^{(\mathrm{X})}(t)$ the reduced density operator for either the atom or the coupling potentials. Tracing out the coupling potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$ degrees of freedom $\hat{\rho}_{\mathrm{A}}^{(\mathrm{X},\xi)}(t) = \mathrm{Tr}_{1,2}\{\hat{\rho}_{\xi}^{(\mathrm{X})}(t)\}$ we obtain the reduced 3×3 atomic density matrix $\hat{\boldsymbol{\rho}}_{\mathrm{A}}^{(\mathrm{X},\xi)}(t)$, elements of which are given by

$$\{\hat{\boldsymbol{\rho}}_{\mathrm{A}}^{(\mathrm{X},\xi)}(t)\}_{jk} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\langle n_1, n_2|\{\hat{\boldsymbol{\rho}}_{\xi}^{(\mathrm{X})}(t)\}_{jk}|n_1, n_2\rangle \tag{34}$$

with $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$. In the same way, the coupling potentials' reduced density operator, obtained from $\hat{\rho}_{\mathrm{P}}^{(\mathrm{X},\xi)}(t) = \mathrm{Tr}_{\mathrm{A}}\{\hat{\rho}_{\xi}^{(\mathrm{X})}(t)\}$, gives

$$\begin{aligned}\hat{\rho}_{\mathrm{P}}^{(\mathrm{X},\xi)}(t) &= \{\hat{\boldsymbol{\rho}}_{\xi}^{(\mathrm{X})}(t)\}_{11} + \{\hat{\boldsymbol{\rho}}_{\xi}^{(\mathrm{X})}(t)\}_{22} + \{\hat{\boldsymbol{\rho}}_{\xi}^{(\mathrm{X})}(t)\}_{33} \\ &= \frac{1}{\mathcal{N}}\{|\mathcal{C}_{\xi}^{(\mathrm{X})}(t)\rangle\langle\mathcal{C}_{\xi}^{(\mathrm{X})}(t)| + |\mathcal{D}_{\xi}^{(\mathrm{X})}(t)\rangle\langle\mathcal{D}_{\xi}^{(\mathrm{X})}(t)| + |\mathcal{E}_{\xi}^{(\mathrm{X})}(t)\rangle\langle\mathcal{E}_{\xi}^{(\mathrm{X})}(t)|\}.\end{aligned} \tag{35}$$

3.2. The atomic level occupation and level transition probabilities

Note that since the probability of finding the atom on its jth level at time t as a result of the transition from the initial atomic state $|3\rangle_{\mathrm{A}}$ can be defined by the formula

$$\begin{aligned}\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t) \equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\left|\langle\psi_{\xi}^{(\mathrm{X})}(t)|n_1, n_2; j\rangle\right|^2 \qquad &\text{where} \\ |n_1, n_2; j\rangle \equiv |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes |j\rangle_{\mathrm{A}}, &\end{aligned} \tag{36}$$

then if we use the expressions (23), (29) and (34) we can show that

$$\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t) = \{\hat{\boldsymbol{\rho}}_{\mathrm{A}}^{(\mathrm{X},\xi)}(t)\}_{jj} \qquad \text{with} \quad j = 1, 2, 3. \tag{37}$$

As shown in [11], we have $\hat{B}_{+}^{(k)}|n\rangle_k = \sqrt{e_{n+1}^{(k)}}|n+1\rangle_k$ and $\hat{B}_{-}^{(k)}|n\rangle_k = \sqrt{e_{n-1}^{(\alpha)} + R_k(a_0^{(k)})}|n-1\rangle_k$. Using these relations, equations (26), (30), (32), (34) and the data of table 1 to obtain the expressions for the diagonal elements of the atoms' reduced density matrix we show that

$$\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(j;t) \tag{38}$$

where the sum contribution terms are obtained by

$$\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(1;t) = P_{n_1n_2}C_{n_1n_2}^{(\mathrm{X},\xi,1)}f_{\mathrm{C}}^2\left(\Theta_{n_1n_2}^{(\mathrm{X},\xi)}t\right) \tag{39}$$

$$\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(2;t) = P_{n_1n_2}C_{n_1n_2}^{(\mathrm{X},\xi,2)}\left|f_{\mathrm{S}}\left(\Theta_{n_1n_2}^{(\mathrm{X},\xi)}t\right)\right|^2 \tag{40}$$

$$\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(3;t) = P_{n_1n_2}\left[1 + C_{n_1n_2}^{(\mathrm{X},\xi,2)}f_{\mathrm{C}}\left(\Theta_{n_1n_2}^{(\mathrm{X},\xi)}t\right)\right]^2 \tag{41}$$

Table 2. Expansion factors for the probabilities $\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t)$ and $\mathcal{P}_{\mathrm{P}}^{(\mathrm{X},\xi)}(n_1,n_2;t)$.

Configuration	$\mathrm{X}=\Xi$	$\mathrm{X}=\Lambda$	$\mathrm{X}=\mathrm{V}$
$C_{nm}^{(\mathrm{X},\mathrm{U},1)}$	$e_n^{(1)}e_m^{(2)}$	$e_n^{(1)}\left(e_m^{(2)}+R_{20}\right)$	$\left(e_n^{(1)}+R_{10}\right)e_m^{(2)}$
$C_{nm}^{(\mathrm{X},\mathrm{U},2)}$	$e_n^{(1)}$	$e_n^{(1)}$	$e_n^{(1)}+R_{10}$
$D_{nm}^{(\mathrm{X},\mathrm{U},1,1)}$	$\left(e_n^{(1)}-e_1^{(1)}\right)e_n^{(1)}e_m^{(2)}$	$\left(e_n^{(1)}-e_1^{(1)}\right)e_n^{(1)}\left(e_m^{(2)}+R_{20}\right)$	$\left(e_n^{(1)}+R_{10}\right)^2e_m^{(2)}$
$D_{nm}^{(\mathrm{X},\mathrm{U},1,2)}$	$e_n^{(1)}e_m^{(2)}\left(e_m^{(2)}-e_1^{(2)}\right)$	$e_n^{(1)}\left(e_m^{(2)}+R_{20}\right)^2$	$\left(e_n^{(1)}+R_{10}\right)e_m^{(2)}\left(e_m^{(2)}-e_1^{(2)}\right)$
$D_{nm}^{(\mathrm{X},\mathrm{U},2,1)}$	$e_n^{(1)}\left(e_n^{(1)}-e_1^{(1)}\right)$	$e_n^{(1)}\left(e_n^{(1)}-e_1^{(1)}\right)$	$\left(e_n^{(1)}+R_{10}\right)^2$
$D_{nm}^{(\mathrm{X},\mathrm{U},2,2)}$	$e_n^{(1)}e_m^{(2)}$	$e_n^{(1)}e_m^{(2)}$	$\left(e_n^{(1)}+R_{10}\right)e_m^{(2)}$
$D_{nm}^{(\mathrm{X},\mathrm{N},1,1)}$	$\left(e_n^{(1)}-e_1^{(1)}\right)\left(e_n^{(1)}e_m^{(2)}\right)^2$	$\left(e_n^{(1)}-e_1^{(1)}\right)\left[e_n^{(1)}(e_m^{(2)}+R_{20})\right]^2$	$\left(e_n^{(1)}+R_{10}\right)^3\left(e_m^{(2)}\right)^2$
$D_{nm}^{(\mathrm{X},\mathrm{N},1,2)}$	$\left(e_m^{(2)}-e_1^{(2)}\right)\left(e_n^{(1)}e_m^{(2)}\right)^2$	$\left(e_n^{(1)}\right)^2\left(e_m^{(2)}+R_{20}\right)^3$	$\left[\left(e_n^{(1)}+R_{10}\right)e_m^{(2)}\right]^2\left(e_m^{(2)}-e_1^{(2)}\right)$
$D_{nm}^{(\mathrm{X},\mathrm{N},2,1)}$	$\left(e_n^{(1)}\right)^2\left(e_n^{(1)}-e_1^{(1)}\right)$	$\left(e_n^{(1)}\right)^2\left(e_n^{(1)}-e_1^{(1)}\right)$	$\left(e_n^{(1)}+R_{10}\right)^3$
$D_{nm}^{(\mathrm{X},\mathrm{N},2,2)}$	$\left(e_n^{(1)}\right)^2e_m^{(2)}$	$\left(e_n^{(1)}\right)^2e_m^{(2)}$	$\left(e_n^{(1)}+R_{10}\right)^2e_m^{(2)}$
$\Theta_{nm}^{(\mathrm{X},\xi)}$	$\vartheta_{nm}^{(\xi,0)}$	$\vartheta_{nm}^{(\xi,2)}$	$\vartheta_{nm}^{(\xi,1)}$

Observation: $C_{nm}^{(\mathrm{X},\mathrm{N},k)}=(C_{nm}^{(\mathrm{X},\mathrm{U},k)})^2$ and $D_{nm}^{(\mathrm{X},\mathrm{N},3,k)}=(D_{nm}^{(\mathrm{X},\mathrm{U},3,k)})^2=(e_n^{(1)})^2$, for $k=1,2$.

and the weight is given by

$$P_{nm}=p_n^{(1)}p_m^{(2)}\Bigg/\sum_{n=0}^{\infty}p_n^{(1)}\sum_{m=0}^{\infty}p_m^{(2)}\qquad\text{with}\qquad p_j^{(k)}=\left|b_j^{(k)}\right|^2,\qquad k=1\text{ or }2. \tag{42}$$

The other factors are presented in table 2.

The function arguments are defined as

$$\vartheta_{nm}^{(\mathrm{U},0)}=\sqrt{e_n^{(1)}+e_m^{(2)}}\qquad \vartheta_{nm}^{(\mathrm{U},1)}=\sqrt{e_n^{(1)}+R_{10}+e_m^{(2)}}\qquad \vartheta_{nm}^{(\mathrm{U},2)}=\sqrt{e_n^{(1)}+e_m^{(2)}+R_{20}}; \tag{43}$$

$$\begin{aligned}&\vartheta_{nm}^{(\mathrm{N},0)}=\sqrt{\left(e_n^{(1)}\right)^2+\left(e_m^{(2)}\right)^2},\qquad \vartheta_{nm}^{(\mathrm{N},1)}=\sqrt{\left(e_n^{(1)}+R_{10}\right)^2+\left(e_m^{(2)}\right)^2},\\ &\vartheta_{nm}^{(\mathrm{N},2)}=\sqrt{\left(e_n^{(1)}\right)^2+\left(e_m^{(2)}+R_{20}\right)^2}.\end{aligned} \tag{44}$$

On the other hand, by using the probability $\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t)$ of finding the atom on its jth level we obtain the rate of transition to level j at time t as $\mathcal{R}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t)\equiv \mathrm{d}\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t)/\mathrm{d}t$. Therefore, with this definition and equations (39), (40) and (41) we can obtain the transition probability for each atomic level.

3.3. The quantum state probability distributions and the intensity of each coupling potential

The probability of finding the coupling potentials in the quantum states (n_1,n_2) at time t is given by

$$\mathcal{P}_{\mathrm{P}}^{(\mathrm{X},\xi)}(n_1,n_2;t)\equiv\sum_{j=1}^{3}\left|\left\langle\psi_{\xi}^{(\mathrm{X})}(t)\middle|n_1,n_2;j\right\rangle\right|^2\qquad\text{where}\quad |n_1,n_2;j\rangle\equiv|n_1\rangle_1\otimes|n_2\rangle_2\otimes|j\rangle_{\mathrm{A}}. \tag{45}$$

Therefore, if we use expressions (23), (29) and (35) we can show that

$$\mathcal{P}_{\mathrm{P}}^{(\mathrm{X},\xi)}(n_1,n_2;t)=\sum_{j=1}^{3}\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(j;t) \tag{46}$$

where the partial contributions $\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(j;t)$, which are related with the three atomic levels, are given by equations (39), (40) and (41).

Note that if we define the quantities $\hat{N}_k=\hat{\mathcal{H}}_{-}^{(k)}\hat{1}$ for $k=1$ or 2, then their expectation value allows us to understand the global influence of each coupling potential in the dynamic of the coupled system. In the case of the harmonic oscillator, the simplest shape-invariant potential, these observables are related with the mean number of photons for the radiation modes 1 and 2 in a given cavity where the atom is immerse. Therefore, in our general case, if we use (24), (26), (30) and (32) we can show that

$$\left\langle\hat{N}_k^{(\mathrm{X},\xi)}(t)\right\rangle=\sum_{j=1}^{3}\mathcal{N}_j^{(\mathrm{X},\xi,k)}(t) \tag{47}$$

where the partial contributions $\mathcal{N}_j^{(\mathrm{X},\xi,k)}(t)$ are obtained by expressions

$$\mathcal{N}_1^{(\mathrm{X},\xi,k)}(t)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1n_2}D_{n_1n_2}^{(\mathrm{X},\xi,1,k)}f_{\mathrm{C}}^2\left(\Theta_{n_1n_2}^{(\mathrm{X},\xi)}t\right) \tag{48}$$

$$\mathcal{N}_2^{(\mathrm{X},\xi,k)}(t)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1n_2}D_{n_1n_2}^{(\mathrm{X},\xi,2,k)}\left|f_{\mathrm{S}}\left(\Theta_{n_1n_2}^{(\mathrm{X},\xi)}t\right)\right|^2 \tag{49}$$

$$\mathcal{N}_3^{(\mathrm{X},\xi,k)}(t)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}P_{n_1n_2}e_{n_k}^{(k)}\left[1+D_{n_1n_2}^{(\mathrm{X},\xi,3,k)}f_{\mathrm{C}}\left(\Theta_{n_1n_2}^{(\mathrm{X},\xi)}t\right)\right]^2. \tag{50}$$

The expansion coefficients $D_{nm}^{(\mathrm{X},\xi,j,k)}$ for each case are presented in table 2.

4. Application for a pair of shape-invariant potentials

As a specific example of our generalized formalism we consider the case of a parasupersymmetric coupled system built with a three-level atom, a Pöschl–Teller and a self-similar potential. The Pöschl–Teller potential [24], closely related to several other potentials widely used in molecular and solid-state physics, has the partner potentials $V_1^{(\pm)}(x)$ in (3) obtained with the superpotentials [25] $W_1\left(x,a_1^{(1)}\right)=\sqrt{\hbar\Omega}\left\{\left(a_1^{(1)}+\gamma\right)\tan\rho x-\left(a_1^{(1)}-\gamma\right)\cot\rho x\right\}$ where $a_1^{(1)}$, ρ and γ are the real constants. The remainders [26] in the shape invariance condition are given by $R_1\left(a_n^{(1)}\right)=4\eta\left[2a_n^{(1)}+\eta\right]$, with the potential parameters related by the translation $a_n^{(1)}=a_1^{(1)}+(n-1)\eta,\forall n\in\mathbb{Z}$, where $\eta=\rho\sqrt{\hbar/(2M\Omega)}$. Inserting these results into (27) we can prove that

$$e_n^{(1)}=4\eta^2n(n+\nu),\qquad\text{yielding}\qquad\prod_{s=0}^{n-1}\sqrt{e_n^{(1)}-e_s^{(1)}}=(2\eta)^n\sqrt{\frac{\Gamma(n+1)\Gamma(2n+\nu)}{\Gamma(n+\nu)}} \tag{51}$$

with $\nu=2a_1^{(1)}/\eta$. Note that in this case $R_{10}\equiv R_1\left(a_0^{(1)}\right)=4\eta\left[2a_1^{(1)}-\eta\right]$.

On the other hand, the self-similar potentials stand for a class of shape-invariant and reflectionless potentials given by an infinite chain of partner potentials $V_{\pm}^{(k)}(y)$, $(k = 0, 1, 2, \ldots)$, the associated superpotentials for which satisfy the self-similar *ansatz* $W_k(y) = q^k W(q^k y)$, with $0 < q < 1$. These sets of partner potentials $V_{\pm}^{(k)}(y)$ have an infinite number of bound states and their parameters related by a scaling [27]: $a_m^{(2)} = a_1^{(2)} q^{m-1}, \forall m \in \mathbb{Z}$. Working with this kind of potential it is possible to get the Rosen–Morse, harmonic oscillator and Pöschl–Teller potentials as limiting cases [27]. Shape invariance of self-similar potentials was studied in detail in [7] and, in the simplest case, the remainders in the shape invariance condition are given by $R_2\big(a_m^{(2)}\big) = c a_m^{(2)}$, where c is a constant. Using this result in (27) we can prove that

$$e_m^{(2)} = \left(\frac{1-q^m}{1-q}\right) R_2\big(a_1^{(2)}\big), \qquad \text{yielding}$$

$$\prod_{s=0}^{m-1} \sqrt{e_m^{(2)} - e_s^{(2)}} = \left[\frac{R_2\big(a_1^{(2)}\big)}{1-q}\right]^{m/2} q^{m(m-1)/4} \sqrt{(q;q)_m} \tag{52}$$

where the q-shifted factorial $(q;q)_m$ is defined as $(p;q)_0 = 1$ and $(p;q)_m = \prod_{k=0}^{m-1}(1-pq^k)$ with $m \in \mathbb{Z}$. In this case we have $R_{20} \equiv R_2\big(a_0^{(2)}\big) = R_2\big(a_1^{(2)}\big)\big/q$.

To obtain the coherent state of the Pöschl–Teller potential, we define the generalizing functional with the form $\mathcal{Z}_{j_1}^{(1)} = \sqrt{g\big(a_1^{(1)}; 4, -4\eta\big) g\big(a_1^{(1)}; 4, 2\eta\big)}$ where the auxiliary function is given by $g\big(a_n^{(1)}; c, d\big) = c a_n^{(1)} + d$. It follows from the potential parameters $a_n^{(1)}$ translation relation that

$$\prod_{s=0}^{n-1} \mathcal{Z}_{j_1+s}^{(1)} = (2\eta)^n \sqrt{\frac{\Gamma\,(2n+\nu)}{\Gamma\,(\nu)}} \tag{53}$$

and using (51) and (53) into (28) we find for the coherent state expansion coefficients

$$b_n^{(1)} = z_1^n \sqrt{\frac{\Gamma(n+\nu)}{\Gamma(\nu)\Gamma(n+1)}}. \tag{54}$$

In the case of the coherent state for the self-similar potential, we assume the simple choice

$$\mathcal{Z}_{j_2}^{(2)} = R_2\big(a_1^{(2)}\big), \qquad \text{yielding} \qquad \prod_{s=0}^{m-1} \mathcal{Z}_{j_2+s}^{(2)} = \big[R_2\big(a_1^{(2)}\big)\big]^m q^{m(m-1)/2}. \tag{55}$$

Substituting this result and (52) into equation (28) we find in this case

$$b_m^{(2)} = \frac{q^{m^2/4}}{\sqrt{(q;q)_m}} \chi^m \tag{56}$$

where $\chi = z_2 \sqrt{R_2\big(a_1^{(2)}\big)(1-q)\big/\sqrt{q}}$. Using these results for $b_n^{(1)}$ and $b_m^{(2)}$, the identity

$$\frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\nu)}{\Gamma(n+1)} |z|^{2n} = (1-|z|^2)^{-\nu}, \tag{57}$$

valid when $|z| < 1$, and the definition of the one-parameter family of q-exponential functions [28, 29]

$$E_q^{(\mu)}(x) = \sum_{m=0}^{\infty} \frac{q^{\mu m^2}}{(q;q)_m} x^m \qquad \text{with} \quad \mu \in \mathbb{R} \tag{58}$$

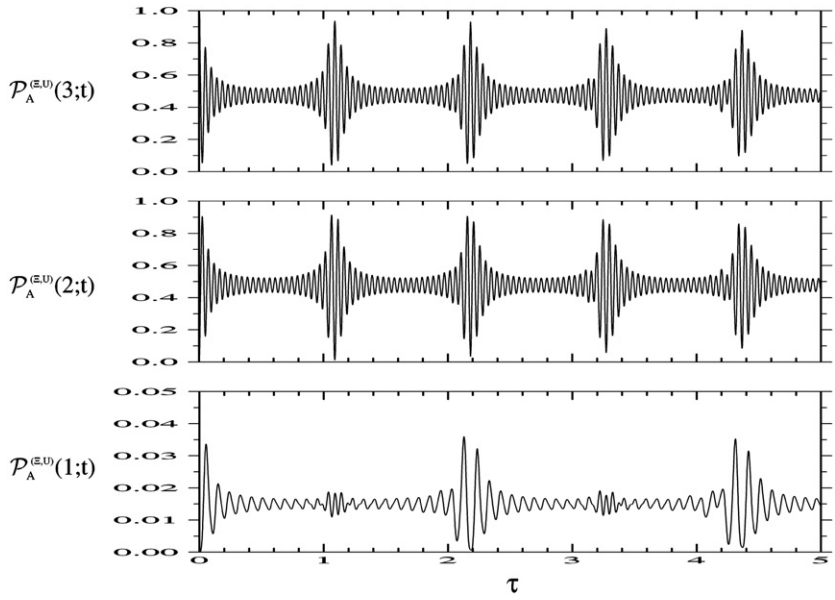

Figure 1. Time evolution of the atomic level occupation probabilities $\mathcal{P}_A^{(\Xi,U)}(j;t)$, with $j = 1, 2, 3$. The set of constants used is given in the text.

we find for the weight in (42) the expression

$$P_{nm} = \frac{(1-|z_1|^2)^\nu}{\Gamma(\nu)E_q^{(1/2)}(|\chi|^2)}\frac{\Gamma(n+\nu)q^{m^2/2}}{\Gamma(n+1)(q;q)_m}|z_1|^{2n}\,|\chi|^{2m}\,. \tag{59}$$

With this set of factors defined and the expressions given in table 2 we can calculate the whole set of observables we introduced in this study.

In figures 1–3 we have plotted the atomic level occupation probabilities $\mathcal{P}_A^{(X,U)}(j;t)$ in terms of the time variable $\tau = gt/\pi$ for the three types of atomic level configurations (X = Ξ, Λ, V). We used in these calculations, and in all other following calculations to be presented, the values $|z_1|^2 = 0.80$, $\eta^2 = 0.2$, $\nu = 20$, $|z_2|^2 = 20$, $R_2\big(a_1^{(2)}\big) = 1$ and $q = 0.80$ which were chosen to make evident the collapse and revival phenomena in these quantities. To understand the time behaviour of the occupation probabilities $\mathcal{P}_A^{(X,U)}(j;t)$ we observe that each term in the double sums (39), (40) and (41) has a different frequency, and as the time increases they become uncorrelated and interfere destructively, causing a collapse $\big[\mathcal{P}_A^{(X,U)}(j;t) \approx 0\big]$. The discrete character of the double sum over the quantum states in the coherent states ensures that, after some finite time, all the oscillating terms come back almost in phase with each other, restoring the coherent oscillations and creating periodic revivals (periodic packets of finite $\mathcal{P}_A^{(X,U)}(j;t)$ oscillations). However, as the frequencies are not necessarily integers and thus may be incommensurate, the re-phasing is not perfect and the revivals get broader and broader. In our systems, the presence of two coupling potentials is responsible for the appearance of the double sum in the expression of $\mathcal{P}_A^{(X,U)}(j;t)$ that opens a great number of the new re-phasing possibilities causing the appearance of new interesting characteristics in the revival events. The periodic behaviour of the circular functions in time, the form of its argument and its dependence on the coupling potentials $e_{n_k}^{(k)}$-factors define the form and the periodicity of these events in $\mathcal{P}_A^{(X,U)}(j;t)$. Comparing the three atomic level occupation probabilities $\mathcal{P}_A^{(X,U)}(j;t)$ for a given configuration X (Ξ, Λ or V) we observe

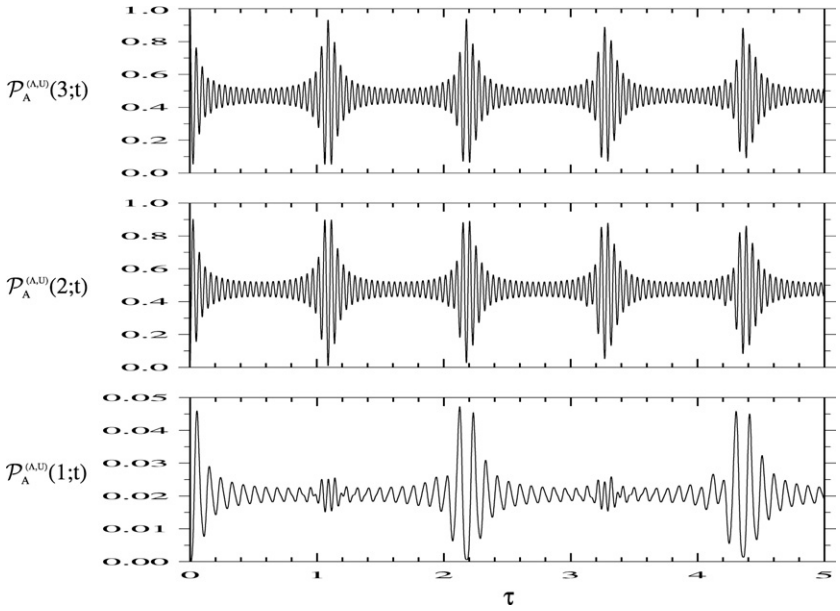

Figure 2. Same as figure 1 for the atomic level occupation probabilities $\mathcal{P}_A^{(\Lambda,U)}(j;t)$, with $j = 1, 2, 3$.

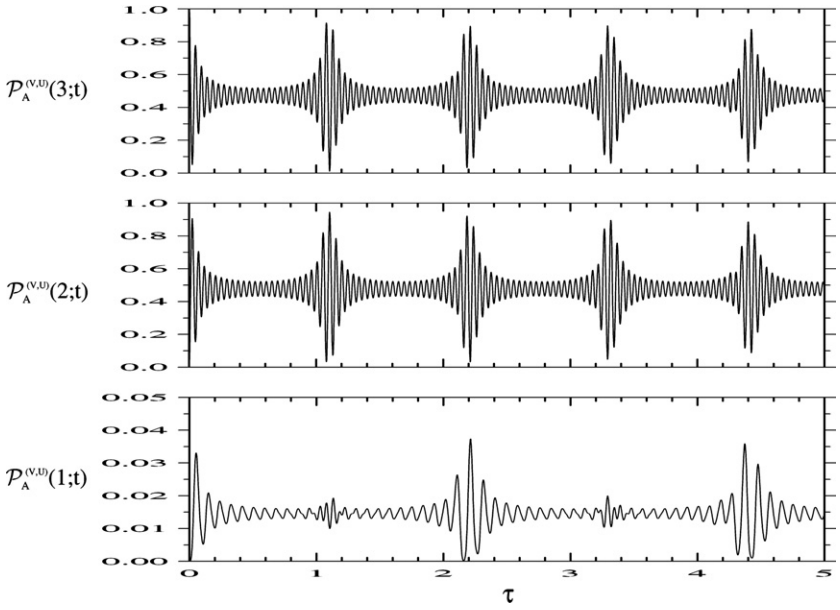

Figure 3. Same as figure 1 for the atomic level occupation probabilities $\mathcal{P}_A^{(V,U)}(j;t)$, with $j = 1, 2, 3$.

that the probability of the atomic level $j = 1$ is considerable smaller than the other two probabilities (for $j = 2$ and 3), whose Rabi oscillations are in phase opposition. Besides that

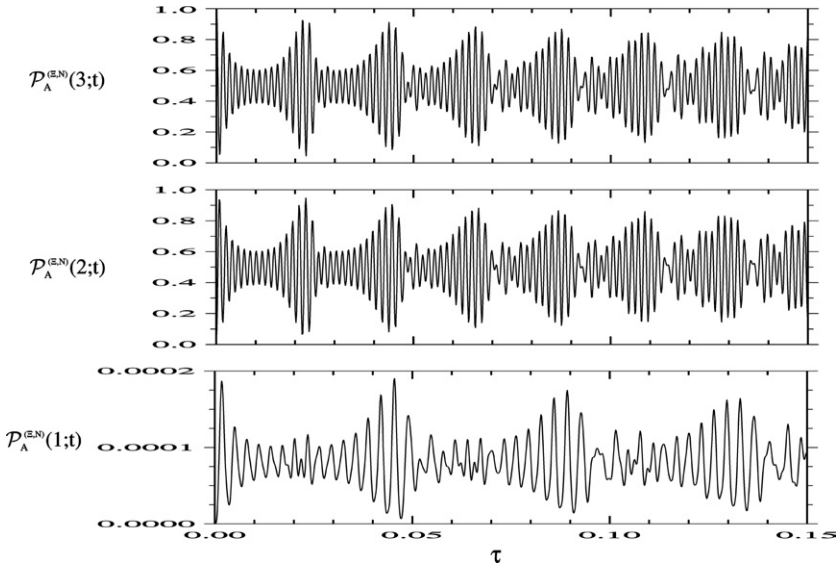

Figure 4. Same as figure 1 for the atomic level occupation probabilities $\mathcal{P}_A^{(\Xi,N)}(j;t)$, with $j = 1, 2, 3$.

its Rabi frequency is almost one-half of the Rabi frequency for the other two ($\nu \approx 27/\tau$). Inside a given atomic level configuration, the probabilities for the three atomic levels show time coincident revival events with period of $\tau_R \approx 1.1\tau$. However, in the case of the atomic level $j = 1$, the revival events of even order are almost suppressed, showing a very reduced structure when compared with the odd order ones. Looking at the envelope of the last revival events we can recognize the asymmetric pattern, raised from the Pöschl–Teller potential influence [13–15]. On the other hand, because of the presence of the self-similar potential in the system, we observe in all of the figures the absence of a complete collapse in the probabilities $\mathcal{P}_A^{(X,U)}(j;t)$, since oscillations of very small amplitude survive where we would observe the complete collapse of $\mathcal{P}_A^{(X,U)}(j;t)$. As a final observation about these first three figures, we note that the results obtained with the three types of atomic level configurations are very close. Because of the small values, only for $\mathcal{P}_A^{(X,U)}(3;t)$ the differences are more visible. The reason for these close results is understood when we compare the expressions of the factors $C_{nm}^{(X,U,k)}$ and $\Theta_{nm}^{(X,\xi)}$ for each type of configuration in table 2, and observe their differences.

Figures 4–6 are the intensity-dependent interaction version ($\xi = N$) of the three first figures. Almost all observations cited in the usual interaction case ($\xi = U$), are still valid in this second situation. However, some additional facts to consider in this case are (i) the smaller values for the revival event period ($\tau_R \approx 0.02\tau$) and for the probabilities $\mathcal{P}_A^{(X,N)}(1;t)$, (ii) the higher values of the Rabi frequency ($\nu \approx 530/\tau$), (iii) the early manifestation of the Pöschl–Teller potential action on the revival events' envelope, (iv) the weak collapse characterization in the probabilities $\mathcal{P}_A^{(X,N)}(j;t)$, which result in a strong revival events' superposition for longer times.

To understand the action of the coupling on the systems related with the shape-invariant potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$, we show in figures 7–9 the decimal logarithm of the expectation values $\langle \hat{N}_k^{(X,\xi)}(t) \rangle$ for the intensity of each coupling potential ($k = 1$ and 2), considering the three types of atomic level configurations ($X = \Xi, \Lambda, V$) and the two kinds of coupling model studied ($\xi = U, N$). Since the expectation value intensity for the two potentials

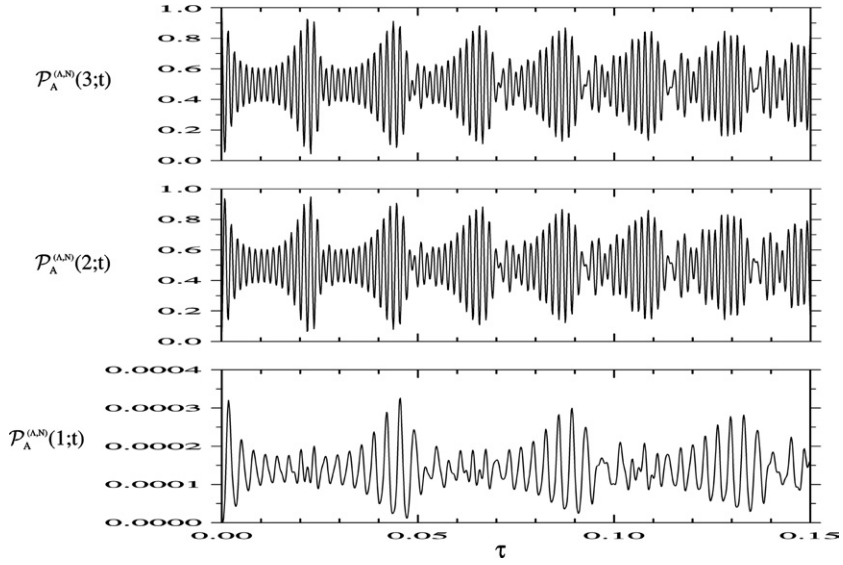

Figure 5. Same as figure 1 for the atomic level occupation probabilities $\mathcal{P}_{\mathrm{A}}^{(\Lambda,\mathrm{N})}(j;t)$, with $j = 1, 2, 3$.

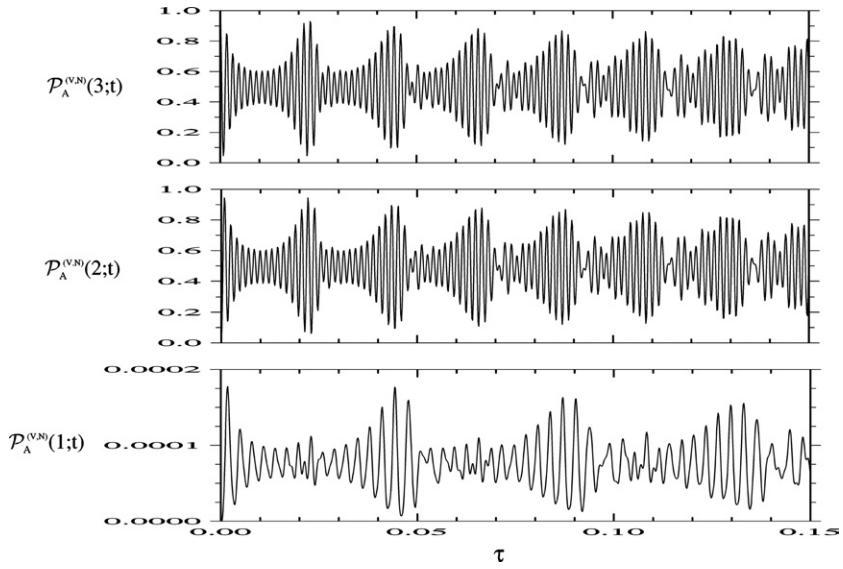

Figure 6. Same as figure 1 for the atomic level occupation probabilities $\mathcal{P}_{\mathrm{A}}^{(\mathrm{V},\mathrm{N})}(j;t)$, with $j = 1, 2, 3$.

are very different, using a logarithmic scale we can compare in a same figure the results obtained for both of them. We observe that for any atomic level configuration and coupling model, the expectation value of the intensity of the Pöschl–Teller potential $\left[\langle \hat{N}_1^{(\mathrm{X},\xi)}(t)\rangle\right]$ is around one thousand times bigger than the expectation value intensity of the self-similar potential $\left[\langle \hat{N}_2^{(\mathrm{X},\xi)}(t)\rangle\right]$. In any atomic level configuration and coupling model, the revival

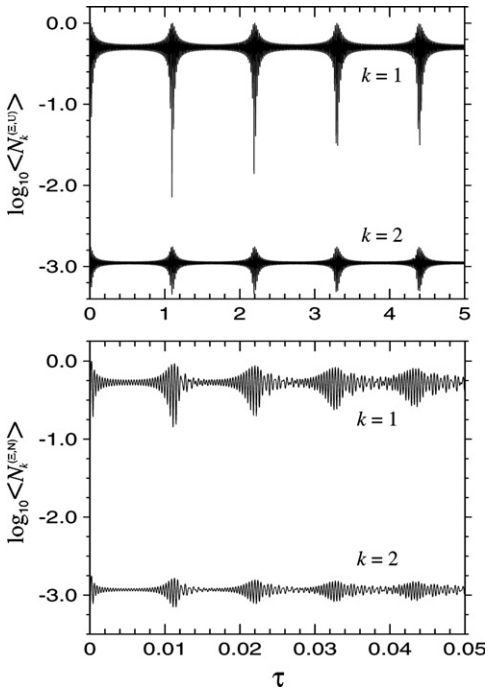

Figure 7. Time evolution of the expectation value of the coupling potential intensities $\langle \hat{N}_k^{(\Xi,\mathrm{U})}(t)\rangle$ and $\langle \hat{N}_k^{(\Xi,\mathrm{N})}(t)\rangle$, with $k = 1, 2$. The set of constants used is the same as of the previous figures.

events in both expectation value intensities are synchronized in time, with periods $\tau_{\mathrm{R}} \approx 1.1\tau$ and $\tau_{\mathrm{R}} \approx 0.011\tau$ for the $\xi = \mathrm{U}$ and the $\xi = \mathrm{N}$ interaction cases, respectively. The Rabi oscillations in $\left\langle \hat{N}_k^{(\mathrm{X,U})}(t)\right\rangle$ have frequency $\nu \approx 50/\tau$ and show revival events with an almost symmetric envelope pattern. On the other hand, in the case of $\left\langle \hat{N}_k^{(\mathrm{X,N})}(t)\right\rangle$ those oscillations have frequency $\nu \approx 2800/\tau$ and the revival events have asymmetric envelope pattern. Again, the differences among the results for the three types of atomic level configurations are very small. As in the case of the atomic level population, the collapse events show an incomplete pattern, where small amplitude oscillations survive. To close the observations about these figures, if we compare them with those for the atomic level populations we can conclude that the revival events in $\left\langle \hat{N}_k^{(\mathrm{X,U})}(t)\right\rangle$ have the same period $\tau_{\mathrm{R}} \approx 1.1\tau$ presented in $\mathcal{P}_{\mathrm{A}}^{(\mathrm{X,U})}(j;t)$. However, in the case of $\left\langle \hat{N}_k^{(\mathrm{X,N})}(t)\right\rangle$, such revival events have period $\tau_{\mathrm{R}} \approx 0.011\tau$, that is around one-half the value for the revival period in $\mathcal{P}_{\mathrm{A}}^{(\mathrm{X,N})}(j;t)$.

Yet to see how the parasupersymmetric coupled system evolves dynamically, we show in figures 10 and 11 the distribution of probability $\mathcal{P}_{\mathrm{P}}^{(\Xi,\mathrm{U})}(n_1,n_2;t)$ of finding the coupling potentials in the quantum states (n_1,n_2), as well as its atomic level partial contributions $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(j;t)$, at times $\tau = 10/\pi$ and $\tau = 20/\pi$, respectively. Because of the wave state $\left|\psi_{\mathrm{U}}^{\Xi}(t)\right\rangle$ normalization condition and the temporal stability of the coupling potential coherent states $\left|z;a_r^{(k)}\right\rangle$, we must have the distribution probability $\mathcal{P}_{\mathrm{P}}^{(\Xi,\mathrm{U})}(n_1,n_2;t)$ constant in time. Therefore, its form presented in figures is the same as given in $\tau = 0$, which corresponds to form (59) of the weight $P_{n_1n_2}$. Due to the very different forms of the potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$ such a distribution of probability shows a different behaviour in relation to the quantum numbers n_1 and n_2, with a very asymmetric distribution ranges. In this case, we

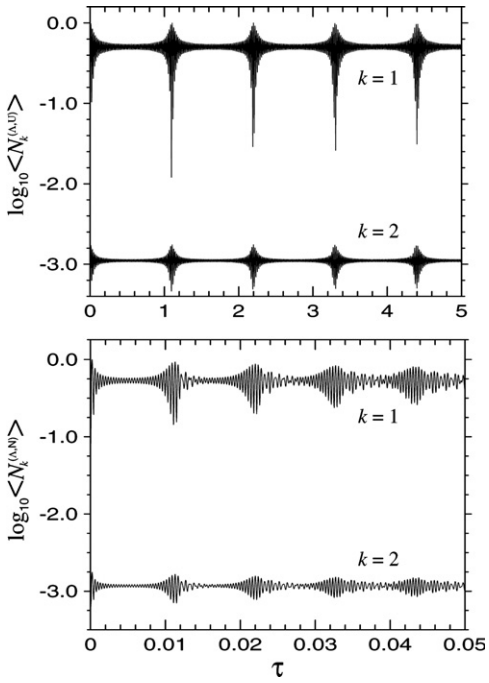

Figure 8. Same as figure 7 for the expectation value of the coupling potential intensities $\langle \hat{N}_k^{(\Lambda,\mathrm{U})}(t)\rangle$ and $\langle \hat{N}_k^{(\Lambda,\mathrm{N})}(t)\rangle$, with $k = 1, 2$.

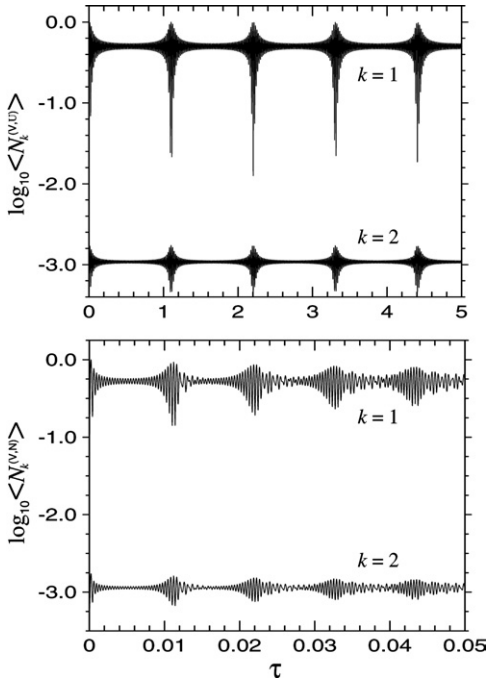

Figure 9. Same as figure 7 for the expectation value of the coupling potential intensities $\langle \hat{N}_k^{(\mathrm{V},\mathrm{U})}(t)\rangle$ and $\langle \hat{N}_k^{(\mathrm{V},\mathrm{N})}(t)\rangle$, with $k = 1, 2$.

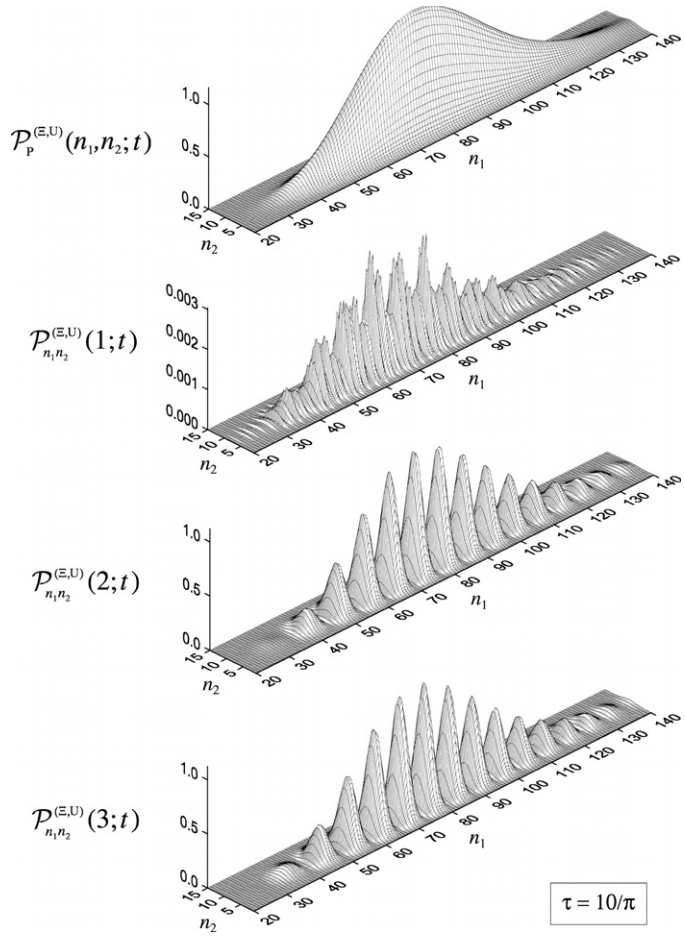

Figure 10. Distributions of probabilities $\mathcal{P}_{\mathrm{P}}^{(\Xi,\mathrm{U})}(n_1, n_2; t)$ and $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(j; t)$, $(j = 1, 2, 3)$, of finding the coupling potentials in the quantum state (n_1, n_2) at $\tau = 10/\pi$. The set of constants used is the same as of the previous figures.

observe that $P_{n_1n_2}$ shows a broader hill pattern in relation to n_1 and a sharper hill pattern in relation to n_2. About the partial contributions, because of the initial condition of the parasupersymmetric coupled system, we must have $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(3; 0) = \mathcal{P}_{\mathrm{P}}^{(\Xi,\mathrm{U})}(n_1, n_2; t)$ and $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(1; 0) = \mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(2; 0) = 0$. It can be seen clearly from figures that when time evolves and the interaction among the subsystems is turned on, the partial distributions $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(2; t)$ and $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(3; t)$ change their initial forms, assuming the pattern of a *chain of mountains* regularly distributed in n_1 and covered by a surface envelope whose form shows some resemblance with the $P_{n_1n_2}$ surface pattern. The hills in the two distributions are in opposition of phase in relation to n_1. For longer times, the number of hills in each distribution chain increases, making them closer. On the other hand, in its time evolution, the partial distribution $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{U})}(3; t)$ starts from the null value assuming, for shorter times, a confused and irregular chain pattern of hills with short heights (around 1% of the heights in the other two probability amplitudes). However, for longer times the chain of hills pattern assumes a regular and defined pattern, like the other

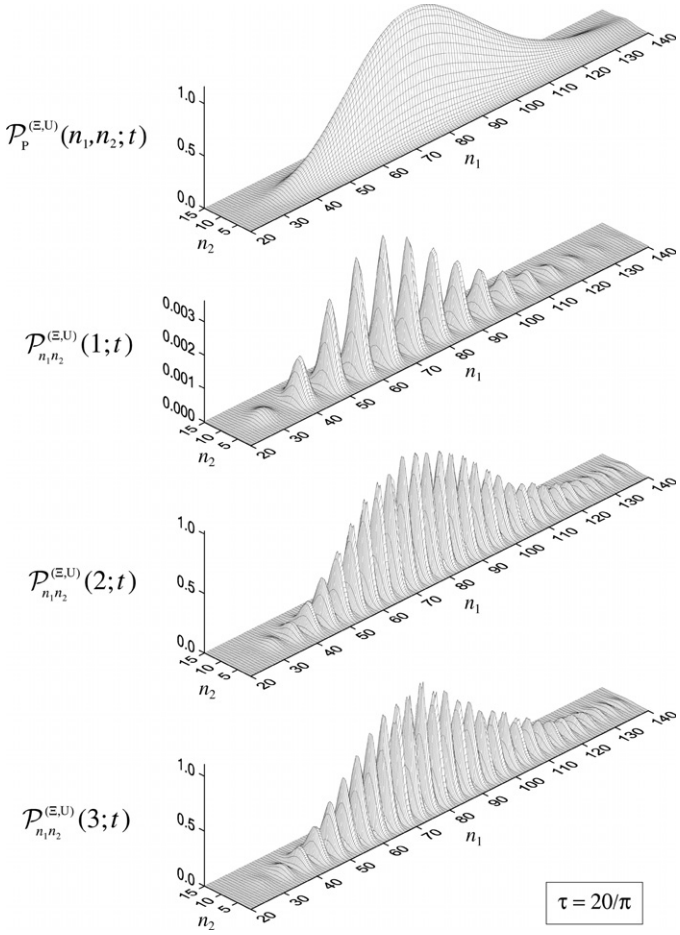

Figure 11. Same as figure 10 at $\tau = 20/\pi$.

two partial probabilities, but with a period in n_1 with almost the double of the value. It should be pointed out that for any time t the constraint imposed by equation (46) remains valid.

Figures 12 and 13 are the intensity-dependent interaction version of figures 10 and 11, where we can see the distribution of probability $\mathcal{P}_{\mathrm{P}}^{(\Xi,\mathrm{N})}(n_1, n_2; t)$ of finding the coupling potentials in the quantum states (n_1, n_2), as well as its atomic level partial contributions $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{N})}(j; t)$, at times $\tau = 0.1/\pi$ and $\tau = 0.2/\pi$, respectively. Note that at the instant $\tau = 0$ the forms of these four distributions are the same as of the usual interaction case. However, in contrast with the previous case, for $\tau > 0$ the partial distributions $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{N})}(k; t)$ now show an irregular chain of hills. Inside of each chain, the hills show different heights and widths and its number increases with time. Another thing to observe is the smaller values of the hill heights in the partial distribution $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{N})}(3; t)$ (around 10^{-6} of the height of the other two). To conclude the observations about this set of figures, note that the presence of the energy factors $e_n^{(k)}$ to the power two in the function arguments $\Theta_{nm}^{(\mathrm{X},\mathrm{N})}$ makes the results obtained in these cases by using the double sums in n_1 and n_2 very irregular and, sometimes, with confusing patterns.

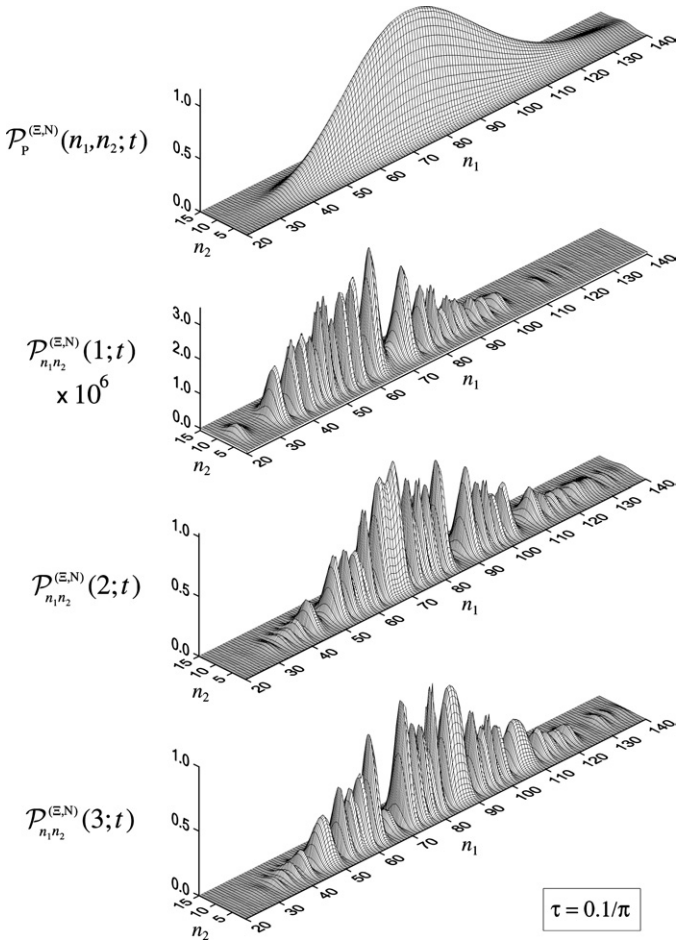

Figure 12. Distributions of probabilities $\mathcal{P}_{\mathrm{P}}^{(\Xi,\mathrm{N})}(n_1, n_2; t)$ and $\mathcal{P}_{n_1n_2}^{(\Xi,\mathrm{N})}(j; t)$, $(j = 1, 2, 3)$, of finding the coupling potentials in the quantum state (n_1, n_2) at $\tau = 0.1/\pi$. The set of constants used is the same as of the previous figures.

Finally, we point out that when we take the limit for the scaling parameter $q \to 1$ and consider that

$$\lim_{q\to 1} e_m^{(2)} \to m R_2\big(a_1^{(2)}\big) \qquad \text{and} \qquad \lim_{q\to 1} \left\{E_q^{(\mu)}[(1-q)\beta]\right\} \to e^{\beta} \tag{60}$$

then we can show that

$$\lim_{q\to 1} P_{nm} \to \frac{(1-|z_1|^2)^{\nu}}{\Gamma(\nu)} \frac{\Gamma(n+\nu)}{\Gamma(n+1)\Gamma(m+1)} |z_1|^{2n} |z_2'|^{2m} \tag{61}$$

where $z_2' = R_2\big(a_1^{(2)}\big) z_2$. Equation (61) corresponds to the weight related with a harmonic oscillator plus Pöschl–Teller potentials [13, 15]. Therefore, when we take the limit $q \to 1$ the results obtained reproduce those corresponding to a parasupersymmetric system coupled with such a pair of potentials.

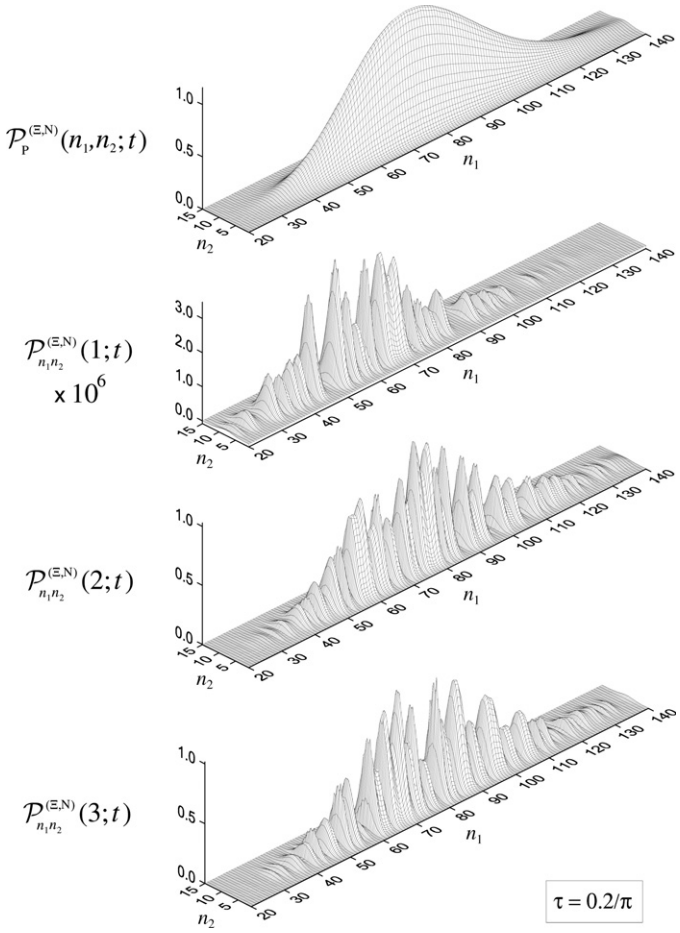

Figure 13. Same as figure 12 at $\tau = 0.2/\pi$.

5. Conclusions

In this paper, we studied the quantum dynamics of a class of bound-state problems which represent a three-level atom coupled with a two-dimensional parasupersymmetric potential system, the independent Cartesian components of which are given by shape-invariant potentials. These exactly soluble and fully quantum-mechanics models were introduced in a previous paper. They represent non-trivial coupled-channel problems which may find applications in molecular, atomic and nuclear physics. Taking into account the three possible configurations of the three energy level of the atom (Ξ-, Λ- and V-type) and two forms of coupling interaction (usual and nonlinear) we used an algebraic formulation to obtain the wave state of the coupled system and the reduced density operator for its subsystems (atom and potentials). The quantum dynamics of this kind of models based on the dipolar and rotating wave approximations strongly depends on the initial conditions of the whole system. In our case, we assume that at time $t = 0$ both potentials are in coherent states and the three-level atom is in one of its eigenstates. By using the density operator of the system, we obtained generalized expressions which give the temporal behaviour of the atomic level occupation

$\mathcal{P}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t)$ and the atomic level transition probabilities $\mathcal{R}_{\mathrm{A}}^{(\mathrm{X},\xi)}(j;t)$ as well as of the quantum states probability distributions $\mathcal{P}_{\mathrm{P}}^{(\mathrm{X},\xi)}(n_1,n_2;t)$ and $\mathcal{P}_{n_1n_2}^{(\mathrm{X},\xi)}(j;t)$ and the expectation value of the intensity of each coupling potential $\left\langle \hat{N}_k^{(\mathrm{X},\xi)}(t)\right\rangle$. As an example, the expressions obtained for these dynamical variables have their behaviour studied for the specific case of a pair of shape-invariant potentials constituted of a Pöschl–Teller and a self-similar potential. The results found in this application exhibit rapid oscillations which periodically collapse and regenerate in different ways depending on the nature of the coupling potentials. Both collapse and revival events are purely quantum effects resulting from the discreteness of the coupling potential spectra and its initial coherent states form. The action of the two shape-invariant potentials gave rise to a multiplicity of new revival events. This quantum phenomenon of decay and regeneration is well known in few restricted cases for the population inversion factor when we have harmonic oscillator coupling potentials. Our results confirm that theoretically the occurrence of this interesting quantum phenomenon is not restricted to the population inversion factor but is shared by the other quantum dynamical variables and is related to the model properties, such as the kind of interaction and the coherent-states associated with the coupling potentials.

Acknowledgments

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